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The gravitational zero mass limit of spin-2 particles

By O. BRULIN and S. HJALMARS

1. Introduction and summary

Relativistic theories for spin-2 particles have been previously investigated by several authors e.g. [1, 2, 3, 4, 5, 6]. The present authors [5] have investigated the spin-2 case, where the components of the wave function of a Dirac equation are the components of a symmetric tensor of second rank and of a vector. This is the simplest spin-2 case in a Dirac equation formalism, where both the coefficients of the equation and the infinitesimal operators of the 4-dimensional orthogonal group R_4 are obtained from the operators of the 5-dimensional orthogonal group R_5. This general formalism and its spin-2 cases have been treated by various authors, e.g. [7, 3, 6], but in these papers it is assumed that the mass term of the Dirac equation is effectively a constant times the unit matrix. The present authors, on the contrary, used in [5] the most general mass term, compatible with the covariance of the Dirac equation. This gave rise to a more general rest mass spectrum, containing unique rest mass as a special case.

However, the most interesting feature of the general mass term is its importance for the zero mass limit. This is already known from the spin-1 case, [8] and [9] p. 68, where such a general mass term is necessary to obtain Maxwell's equations as a limit of the Dirac equation for the vector meson. It is therefore very probable that a general mass term is also necessary in order to obtain the linearized Einstein equations for the gravitational field as a zero mass limit of a Dirac equation for a spin-2 particle.

In a preliminary communication [10] to the paper [5] the authors mentioned the fact that the investigated simplest spin-2 case has a zero mass limit, where the tensor satisfies the graviton equations in a special gauge. However, in this case the tensor depends upon the vector in such a way that the tensor has the same structure as the arbitrary gauge term, which can be added to the field tensor in the linearized theory of gravitation. Consequently, the simplest case of spin-2 cannot give anything but a flat metric in the linear approximation. This situation is analogous to the simplest case of spin 1, which is the scalar meson, according to the classification [3] by means of the highest eigenvalue of the infinitesimal operators of R_4. Here the wave function contains a field vector, which is the gradient of a field scalar. Such a vector has evidently the same form as the arbitrary gauge term, which can be added to the field vector in the Maxwell zero mass limit of spin 1. It may thus be conjectured that the linearized gravitational theory can be obtained as a suitable zero mass limit of one or the other of the two higher spin-2 cases.

Leaving the other spin-2 case to a forthcoming treatment, we will in the present paper investigate the case, where the components $\psi^{\alpha\beta\gamma}$ of the wave function of the Dirac equation are the components of a 5-dimensional tensor of third rank with one antisymmetric index-pair, (3.9), and with vanishing cyclic permutations, (3.8). As discussed in Section 3 such a tensor has 40 independent components and is irreducible under the affine 5-dimensional group. Now, if the matrix coefficients β_k, λ of a Dirac equation, (2.32), and the operators q_{kl}, (2.33), of R_4 were to be deduced from the operators of R_5 as in [5], the above-mentioned tensor has to be made irreducible under R_5 by imposing further conditions, i.e. the vanishing of the contractions of the tensor. This reduces the number of independent components to 35, giving one of the spin-2 cases, classified by Bhabha [3] and developed in detail by Tsuneto and Fujiwara [6].

However, our investigation has shown that this 35-component Dirac equation, even with the most general mass term, is not able to give the graviton equations as a zero mass limit. In order to obtain a Dirac equation with this property it has proved necessary to construct the matrix coefficients β_k, λ and the operators q_{kl} of R_4 by means of the operators Q_α^β, (2.15), of the affine 5-dimensional group. This construction is performed in Section 2, starting with the construction of a Dirac equation, (2.32), (2.29), (2.16), (2.17), (2.12), covariant under affine 4-dimensional transformations. This equation is restricted to R_4, (2.34), (2.35). The four β_k:s of the Dirac equation (2.32) do not in general form an irreducible set. Nevertheless, it is proved that the Dirac equation is irreducible, provided that the mass term, λ, is of sufficiently general form. By the same arguments as in [11] it is proved also for the present case that the spin of the particle is given by the highest eigenvalue of q_{kl}.

In Section 3 it is shown that our special wave function $\psi^{\alpha\beta\gamma}$ is split under R_4 into a tensor of third rank, (3.10), a symmetric and an antisymmetric tensor of second rank, (3.12), (3.14), two vectors, (3.16), (3.17), and a scalar, (3.18), all tensors having vanishing contractions.

In Section 4 the matrices β_k, (4.3), q_{kl}, (4.5), and the metric form β, (4.8), are represented by the sub-space method of Klein, used previously in [9,5]. Moreover, the most general mass-term λ is constructed, (4.14).

In Section 5 the Dirac equation is written in component form, (5.1)–(5.6). The mass-spectrum is deduced, (5.7), and it is shown that unique mass can be obtained in several ways.

The gravitational zero mass limit is discussed in Section 6. It proves necessary to make a slight generalization of the Dirac equation, (2.32), to a form, (6.1), where the β_k:s are substituted by $\Lambda\beta_k\Lambda$, the matrix Λ, (6.2), being of the same form as the mass term λ, but Hermitian.

By putting some of the arbitrary constants in Λ and λ equal to zero we can get rid of the antisymmetric tensor, (3.14), and one of the vectors, (3.17), as well as of the corresponding components of the Dirac equation. By this procedure we obtain in fact a 30-component Dirac equation. After a suitable disposition of the remaining arbitrary constants the components of this Dirac equation can be identified with the linearized Einstein equations for the gravitational field in empty space and in a general gauge. For this identification we have only to assume the symmetrical part of ψ_{kl5} proportional to the deviations (6.12), (6.17), of the metric fundamental tensor from its normal values and to assume ψ_{klm} proportional to the gravitational superpotentials, (6.11), introduced by Freud [13].

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2. Covariance of a wave equation under the affine group

We consider wave equations of the form (latin indices running from 1 to 4 and $x^4 = ict$):

$$(\eta^k p_k + \vartheta_k p^k - \lambda)\psi = 0, \tag{2.1}$$

where

$$p_k = -i\hbar\,\partial_k = -i\hbar\frac{\partial}{\partial x^k}, \tag{2.2}$$

$$p^k = g^{kl}\,p_l = -i\hbar\,g^{kl}\,\partial_l = -i\hbar\,g^{kl}\frac{\partial}{\partial x^l}. \tag{2.3}$$

Now perform an affine transformation,

$$p^{k'} = a^k_r\,p^r = p^k + \varepsilon^k_r\,p^r + \cdots, \tag{2.4}$$

$$p'_k = b^r_k\,p_r = p_k - \varepsilon^r_k\,p_r + \cdots, \tag{2.5}$$

and a simultaneous transformation of ψ, according to

$$\psi' = Q\psi = (1 + \varepsilon^r_s Q_r{}^s + \cdots)\psi, \tag{2.6}$$

$$[Q_s{}^r, Q_l{}^k] = g_l{}^r Q_s{}^k - g_s{}^k Q_l{}^r. \tag{2.7}$$

In the new coordinate system the wave equation (2.1) takes the form

$$(\eta^k p'_k + \vartheta_k p^{k'} - \lambda')\psi' = 0, \tag{2.8}$$

with

$$\lambda' = Q\lambda Q^{-1}, \tag{2.9}$$

provided that

$$a^k_s Q\eta^s Q^{-1} = \eta^k, \tag{2.10}$$

$$b^r_k Q\vartheta_r Q^{-1} = \vartheta_k. \tag{2.11}$$

The most general form of λ is seen to be any fully contracted expression in η^k, ϑ_k, g^{kl} and g_{kl}:

$$\lambda = g_{rs}g^{pq}\ldots\eta^r\ldots\vartheta_k\ldots\eta^s\ldots\eta^k\ldots\vartheta_p\ldots\vartheta_q\ldots. \tag{2.12}$$

The transformation (2.9) thus means that λ' is the same expression as (2.12) but with g_{kl} and g^{kl} substituted by g'_{kl} and $g^{kl'}$.

From (2.10) and (2.11) we obtain

$$[\eta^k, Q_l{}^m] = g_l{}^k\eta^m, \tag{2.13}$$

$$[\vartheta_k, Q_l{}^m] = -g_k{}^m\vartheta_l. \tag{2.14}$$

Also for affine transformations we can consider the 4-dimensional space as a subspace of a 5-dimensional one. The Q_α^β:s (Greek indices running from 1 to 5) for the 5-dimensional affine transformations satisfy

$$[Q_\alpha^\beta, Q_\delta^\gamma] = g_\delta^\beta Q_\alpha^\gamma - g_\alpha^\gamma Q_\delta^\beta. \tag{2.15}$$

If we put

$$\eta^k = -iQ_5{}^k, \tag{2.16}$$

$$\vartheta_k = iQ_k{}^5, \tag{2.17}$$

$$\nu = Q_5{}^5, \tag{2.18}$$

the equations (2.15) contain (2.7), (2.13) and (2.14) as special cases. Moreover we have

$$[\eta^k, \eta^l] = [\vartheta_k, \vartheta_l] = 0, \tag{2.19}$$

$$[\eta^k, \vartheta_l] = g_l{}^k \nu - Q_l^k, \tag{2.20}$$

$$[\nu, Q_k{}^l] = 0, \tag{2.21}$$

$$[\nu, \eta^k] = \eta^k, \tag{2.22}$$

$$[\nu, \vartheta_k] = -\vartheta_k. \tag{2.23}$$

We now assume the set of ψ-components to be transformed irreducibly under the 5-dimensional affine group, which means that the operators Q_α^β form a set of irreducible matrices. Since by (2.15)

$$[Q_\alpha^\alpha, Q_\delta^\gamma] = 0, \tag{2.24}$$

Q_α^α must be a multiple k of the unit matrix:

$$Q_\alpha^\alpha = k \tag{2.25}$$

or

$$Q_r^r = k - \nu. \tag{2.26}$$

By contracting (2.20), we obtain

$$\nu = \frac{1}{5}\{[\eta^r, \vartheta_r] + k\}. \tag{2.27}$$

The equations (2.27) and (2.20) give

$$Q_l^k = [\vartheta_l, \eta^k] + \frac{1}{5}\{[\eta^r, \vartheta_r] + k\} g_l^k. \tag{2.28}$$

Since Q_α^β, i.e. $Q_k{}^l$, η^k, ϑ_k, ν, form an irreducible set, and since $Q_k{}^l$ and ν can be expressed in η^k, ϑ_k by means of (2.27), (2.28), we can infer that η^k, ϑ_k must form an irreducible set.

Let us introduce the matrices

$$\beta^k = \eta^k + g^{kl}\vartheta_l. \tag{2.29}$$

From (2.22), (2.23) and (2.29) we get

$$\eta^k = \frac{1}{2}\{\beta^k + [\nu, \beta^k]\}, \tag{2.30}$$

$$\vartheta_k = \frac{1}{2} g_{kl}\{\beta_l - [\nu, \beta^l]\}. \tag{2.31}$$

These equations show that the set β^k, ν also form an irreducible set.

The wave equation (2.1) can now be written

$$(\beta^k p_k - \lambda)\psi = 0. \tag{2.32}$$

Furthermore, the term λ can be considered as any fully contracted expression in β^k, ν and g_{kl}. It follows from the preceding arguments that (2.32) is irreducible, if the dependence of λ on ν is sufficiently general.

If we restrict ourselves to the orthogonal 4-dimensional group R_4, we have $\varepsilon^r_s = \varepsilon_{rs} = -\varepsilon_{sr}$ and $g_{kl} = \delta_{kl}$ and (2.6) can be written

$$\psi' = Q\psi = \left(1 + \frac{1}{2}\varepsilon^{rs} q_{rs} + \cdots\right)\psi, \tag{2.33}$$

where

$$q_{rs} = Q_r^s - Q_s^r. \tag{2.34}$$

From (2.20) and (2.29) follows

$$q_{kl} = [\beta_k, \beta_l], \tag{2.35}$$

which is the special dependence of the q_{kl}:s upon the β_k:s, adopted e.g. in [7, 3, 6, 5]. However, the present scheme is more general, since the β_k:s of (2.29) are in most cases reducible, contrary to the usual assumption for the matrix coefficients of a relativistic wave equation. As mentioned above, the irreducibility of the equation is secured by a more general type of mass term.

The expressions for energy and spin densities, given by Bhabha [11], are valid in this case too, since the present λ also commutes with the operators q_{kl}. The spin is thus here, as in the usual scheme, given by the highest eigenvalue of q_{kl}.

A special example of the β^k:s (2.29) of the affine case is furnished by the authors' previous generalization of the Kemmer matrices to general relativity [12]. By comparing (2.29) with (47) of [12] we find the following connection:

$$\eta^k = \overset{\circ}{\beta}{}^k \overset{\circ}{\omega}_4, \tag{2.36}$$

$$\vartheta_k = \overset{\circ}{\beta}_k \overset{\circ}{\omega}_5. \tag{2.37}$$

Moreover it can be proved that

$$\nu = \overset{\circ}{\omega}_5, \tag{2.38}$$

showing that ν can here be expressed by means of the β^k:s. This is in agreement with the fact that in this special case the β^k:s alone form an irreducible set.

Another example is given by the generalization to affine transformations of the spin-2 case, previously treated by the authors [5]. This is performed by abandoning the spur condition (2.16) of [5], giving 15-rowed matrices instead of 14-rowed. In this case, however, the Dirac equation gives a homogeneous linear connection between ψ_{55} and $\psi_r{}^r$, which has the same consequences as the condition (2.16) of [5]. Thus, under the restriction to R_4 we obtain component equations, which are essentially the same as (5.1), (5.2) of [5]. Our present generalization gives consequently nothing new for this lowest spin-2 case. However, in the other spin-2 cases, one of which will be treated in this paper, the generalization will give new possibilities.

3. Properties of the wave function

In the present spin-2 case the wave function consists of the components of a certain 5-dimensional tensor of third rank, irreducible under the affine 5-dimensional group. Tensors with this property can be obtained from the primitive characteristic units of the Frobenius algebra for the symmetric group of permutations on three symbols α, β, γ. If the units of such a set operate on the indices of an arbitrary tensor $T_{\alpha\beta\gamma}$ with $D_5 = 125$ components, the tensor is decomposed into irreducible tensors, which transform separately under the affine group.

Sets of primitive characteristic units can be constructed by various procedures from the standard Young tableaux

$$(\alpha\beta\gamma),\quad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix},\quad \begin{pmatrix}\alpha & \beta\\ \gamma & \end{pmatrix},\quad \begin{pmatrix}\alpha & \gamma\\ \beta & \end{pmatrix}.$$

One such procedure gives the usual Young tableau units. However, for the present purpose, it will be more convenient to use a slightly different but equivalent procedure, giving units, which split the tensor $T^{\alpha\beta\gamma}$ in the following irreducible tensors:

$${}^1T^{\alpha\beta\gamma} = \tfrac{1}{6}\,(T^{\alpha\beta\gamma} + T^{\beta\gamma\alpha} + T^{\gamma\alpha\beta} + T^{\gamma\beta\alpha} + T^{\beta\alpha\gamma} + T^{\alpha\gamma\beta}), \tag{3.1}$$

$${}^2T^{\alpha\beta\gamma} = \tfrac{1}{6}\,(T^{\alpha\beta\gamma} + T^{\beta\gamma\alpha} + T^{\gamma\alpha\beta} - T^{\gamma\beta\alpha} - T^{\beta\alpha\gamma} - T^{\alpha\gamma\beta}), \tag{3.2}$$

$${}^3T^{\alpha\beta\gamma} = \tfrac{1}{3}\,(T^{\alpha\beta\gamma} + T^{\gamma\beta\alpha} - T^{\alpha\gamma\beta} - T^{\beta\gamma\alpha}), \tag{3.3}$$

$${}^4T^{\alpha\beta\gamma} = \tfrac{1}{3}\,(T^{\alpha\beta\gamma} + T^{\alpha\gamma\beta} - T^{\gamma\beta\alpha} - T^{\gamma\alpha\beta}). \tag{3.4}$$

The tensor ${}^1T^{\alpha\beta\gamma}$ is symmetric in all index pairs. The formula $D = \begin{pmatrix} s+n-1 \\ n-1 \end{pmatrix}$ for the number of independent components of a completely symmetric tensor of rank s in n dimensions gives ${}^1D_5 = 35$ for the tensor ${}^1T^{\alpha\beta\gamma}$.

Under restriction to the 4-dimensional orthogonal group R_4 the tensor ${}^1T^{\alpha\beta\gamma}$ is split into the following six irreducible tensors with vanishing contractions:

$$\left.\begin{aligned} &{}^1T_{klm} - \tfrac{1}{3}\,(T_r{}^r{}_m\,\delta_{kl} + T_{rl}{}^r\,\delta_{km} + T_{kr}{}^r\,\delta_{lm}),\\ &{}^1T_{kl5} - \tfrac{1}{4}\,{}^1T_r{}^r{}_5\,\delta_{kl},\\ &{}^1T_{kr}{}^r,\ {}^1T_{k5}{}^5,\ {}^1T_{5r}{}^r,\ {}^1T_{555}, \end{aligned}\right\} \tag{3.5}$$

having 16, 9, 4, 4, 1, 1 independent components respectively. According to the formula

$$\left.\begin{aligned} d_4(\lambda_1, \lambda_2) &= 2(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 + 1) \qquad (\lambda_2 \neq 0) \\ d_4(\lambda_1, 0) &= (\lambda_1 + 1)^2 \end{aligned}\right\} \tag{3.6}$$

for the degree $d_4(\lambda_1, \lambda_2)$ of a representation $R_4(\lambda_1, \lambda_2)$ of the group R_4, the above-mentioned tensors are seen to correspond to the representations $R_4(3,0)$, $R_4(2,0)$, $R_4(1,0)$, $R_4(1,0)$, $R_4(0,0)$, $R_4(0,0)$ respectively. Consequently the highest eigenvalue of q_{kl} is 3 and ${}^1T^{\alpha\beta\gamma}$ is thus a case of spin 3.

The tensor ${}^2T^{\alpha\beta\gamma}$ is antisymmetric in all index pairs and the number of independent components is thus the same as the number of independent antisymmetric combinations of the remaining 2 indices, i.e. ${}^2D_5 = 10$.

Under restriction to R_4 ${}^2T^{\alpha\beta\gamma}$ is split into the tensors ${}^2T_{kl5}$ and ${}^2T_{klm}$ with 6 and 4 independent components and corresponding to $R_4(1,1)$ and $R_4(1,0)$ respectively. ${}^2T^{\alpha\beta\gamma}$ thus corresponds to a case of spin 1.

The two tensors ${}^3T^{\alpha\beta\gamma}$ and ${}^4T^{\alpha\beta\gamma}$ have both vanishing sum of cyclic permutations and are moreover antisymmetric in one index pair, ${}^3T^{\alpha\beta\gamma}$ in $\beta\gamma$ and ${}^4T^{\alpha\beta\gamma}$ in $\alpha\gamma$. Since they correspond to equivalent representations, they have the same number of independent components, ${}^3D_5 = {}^4D_5 = \frac{1}{2}(D_5 - {}^1D_5 - {}^2D_5) = 40$. This number can also be checked in the following way. The number of independent combinations of the antisymmetric index pair is 10. Every such combination can be combined with each of the 5 values of the remaining index, giving 50 components. The sum of the cyclic permutations of a tensor with an antisymmetric index pair is a completely antisymmetric tensor, having 10 components, and these put equal to zero give 10 extra conditions. The number of independent components is thus $50 - 10 = 40$, as above.

As will be seen below the equivalent tensors ${}^3T^{\alpha\beta\gamma}$ and ${}^4T^{\alpha\beta\gamma}$ correspond to a certain case of spin 2, and we will choose

$$\psi^{\alpha\beta\gamma} = {}^3T^{\alpha\beta\gamma} \tag{3.7}$$

as the wave function of the spin-2 case to be investigated in this paper. As mentioned above, we have

$$\psi^{\alpha\beta\gamma} + \psi^{\beta\gamma\alpha} + \psi^{\gamma\alpha\beta} = 0, \tag{3.8}$$

$$\psi^{\alpha\beta\gamma} = -\psi^{\alpha\gamma\beta}. \tag{3.9}$$

Under restriction to R_4 the wave function $\psi^{\alpha\beta\gamma}$ is split in the following seven irreducible tensors:

A) A tensor of third rank,

$$A_{klm} = \psi_{klm} - \tfrac{1}{3}(\psi_r{}^r{}_m\,\delta_{kl} + \psi_{rl}{}^r\,\delta_{km}), \tag{3.10}$$

satisfying (3.8), (3.9) and

$$A_{rl}{}^r = A_r{}^r{}_m = A_{krr} = 0. \tag{3.11}$$

This tensor has ${}^Ad_4 = 16$ independent components, corresponding to $R_4(2,1)$.

B) A symmetric tensor of second rank and vanishing contraction,

$$B_{kl} = \tfrac{1}{2}(\psi_{kl5} + \psi_{lk5}) - \tfrac{1}{4}\psi_r{}^r{}_5\,\delta_{kl}, \tag{3.12}$$

satisfying

$$B_{kl} = B_{lk},\ B_r^{\ r} = 0. \tag{3.13}$$

This tensor has ${}^{B}d_4 = 9$ independent components, corresponding to $R_4(2,0)$.

C) An antisymmetric tensor of second rank,

$$C_{kl} = \tfrac{1}{2}(\psi_{kl5} - \psi_{lk5}) = -\tfrac{1}{2}\psi_{5kl}, \tag{3.14}$$

satisfying

$$C_{kl} = -C_{lk},\ C_{rr} = 0. \tag{3.15}$$

This tensor has ${}^{C}d_4 = 6$ independent components, corresponding to $R_4(1,1)$.

D) A vector

$$D_k = \psi_r{}^r{}_k = -\psi_{rk}{}^r, \tag{3.16}$$

with ${}^{D}d_4 = 4$ independent components, corresponding to $R_4(1,0)$.

E) A vector

$$E_k = -\psi_{55k} = \psi_{5k5} \tag{3.17}$$

with ${}^{E}d_4 = 4$ independent components, corresponding to $R_4(1,0)$.

F) A scalar

$$F = \psi_r{}^r{}_5 = -\psi_{r5}{}^r \tag{3.18}$$

with ${}^{F}d_4 = 1$ and corresponding to $R_4(0,0)$.

In the following we will consider the independent A, B, C, D, E, F-components as the independent components of the wave function ψ.

Any component of $\psi_{\alpha\beta\gamma}$ can be expressed in the tensors A_{klm}, B_{kl}, C_{kl}, D_k, E_k, F by means of (3.16)–(3.18) and the following formulae:

$$\psi_{klm} = A_{klm} + \tfrac{1}{3}(D_m\delta_{kl} - D_l\delta_{km}), \tag{3.19}$$

$$\psi_{kl5} = B_{kl} + C_{kl} + \tfrac{1}{4}F\delta_{kl}, \tag{3.20}$$

$$\psi_{5lm} = -2C_{lm}. \tag{3.21}$$

4. Representation of the matrices

Following a procedure, analogous to that in Section 2 of [5], we identify the transformation (2.6) in 5 dimensions with the affine transformation in 5 dimensions of a tensor of third rank:

$$\psi^{\alpha\beta\gamma} + \varepsilon_\sigma^\xi (Q_\xi{}^\sigma\psi)^{\alpha\beta\gamma} = \psi^{\alpha\beta\gamma} + \varepsilon_\sigma^\alpha\psi^{\sigma\beta\gamma} + \varepsilon_\sigma^\beta\psi^{\alpha\sigma\gamma} + \varepsilon_\sigma^\gamma\psi^{\alpha\beta\sigma}. \tag{4.1}$$

From (4.1) follows:

$$(Q_\xi{}^\sigma\psi)^{\alpha\beta\gamma} = g_\xi{}^\alpha\psi^{\sigma\beta\gamma} + g_\xi{}^\beta\psi^{\alpha\sigma\gamma} + g_\xi{}^\gamma\psi^{\alpha\beta\sigma}. \tag{4.2}$$

Now, restricting ourselves to R_4, we obtain by (2.16), (2.17), (2.18), (2.29), (2.34) and (4.2):

$$(\beta_r \psi)_{\alpha\beta\gamma} = i\,(\delta_{r\alpha}\psi_{5\beta\gamma} + \delta_{r\beta}\psi_{\alpha 5\gamma} + \delta_{r\gamma}\psi_{\alpha\beta 5} - \\ - \delta_{5\alpha}\psi_{r\beta\gamma} - \delta_{5\beta}\psi_{\alpha r\gamma} - \delta_{5\gamma}\psi_{\alpha\beta r}), \tag{4.3}$$

$$(\nu \psi)_{\alpha\beta\gamma} = \delta_{5\alpha}\psi_{5\beta\gamma} + \delta_{5\beta}\psi_{\alpha 5\gamma} + \delta_{5\gamma}\psi_{\alpha\beta 5}, \tag{4.4}$$

$$(q_{rs} \psi)_{\alpha\beta\gamma} = \delta_{r\alpha}\psi_{s\beta\gamma} + \delta_{r\beta}\psi_{\alpha s\gamma} + \delta_{r\gamma}\psi_{\alpha\beta s} - \\ - \delta_{s\alpha}\psi_{r\beta\gamma} - \delta_{s\beta}\psi_{\alpha r\gamma} - \delta_{s\gamma}\psi_{\alpha\beta r}. \tag{4.5}$$

Since the 40×40-rowed matrices β_k, ν form an irreducible set, any 40×40-rowed matrix can be expressed as a polynomial $f(\beta_k, \nu)$ in the β_k:s and ν.

In the following the result $f(\beta_k, \nu)\psi$ of the operation on ψ by an operator $f(\beta_k, \nu)$ is obtained in calculating the components $[f(\beta_k, \nu)\psi]_{\alpha\beta\gamma}$ by repeated use of (4.3) and (4.4). Since, in the end, we are interested in the A, B, C, D, E, F-components of $f(\beta_k, \nu)\psi$, we form these components from $[f(\beta_k, \nu)\psi]_{\alpha\beta\gamma}$ by the procedure, indicated by (3.10), (3.12), (3.14), (3.16), (3.17) and (3.18). In the result all components $\psi_{\alpha\beta\gamma}$ of the original wave function ψ are also expressed in the original A, B, C, D, E, F-components by means of (3.16), (3.17), (3.18), (3.19), (3.20), (3.21).

Of course we could also, as in [5], construct explicit expressions for the matrix elements of β_k, ν, q_{kl}, using the special properties (3.8) and (3.9) of the wave function $\psi_{\alpha\beta\gamma}$. We will not perform this procedure here, since, for the purpose of the present paper, we have no need for a detailed knowledge of the special algebra of the matrices.

Since the matrices q_{kl} form a reducible representation of R_4, composed by $R_4(2,1)$, $R_4(2,0)$, $R_4(1,1)$, $R_4(1,0)$ and $R_4(0,0)$, they have the eigenvalues $0, \pm 1, \pm 2$. Thus they satisfy the minimal equation

$$q_{kl}^5 + 5q_{kl}^3 + 4q_{kl} = 0. \tag{4.6}$$

As a consequence of (2.35) and (4.6) the β_k:s satisfy the general algebra of spin-2 matrices, given by Madhava Rao [4]. Moreover, the metric form

$$\beta = \tfrac{1}{3}(2\beta_4^4 - 8\beta_4^2 + 3), \tag{4.7}$$

given by the authors [5] for the simplest spin-2 case is also valid in the present case. In fact, β is a satisfactory metric form, if it satisfies (3.5)–(3.8) of [5]. In proving (3.5)–(3.7) of [5] it is only necessary to use the general spin-2 algebra of Madhava Rao, satisfied also in the present case. The validity of (3.8) of [5] will be proved below in connection with the discussion of the λ-term. From (4.3) and (4.7) we obtain:

$$\tilde{\psi}_{\alpha\beta\gamma} = (\psi^* \beta)_{\alpha\beta\gamma} = \pm \psi^*_{\alpha\beta\gamma}, \tag{4.8}$$

where the minus sign is to be chosen, if and only if an odd number of the indices α, β, γ are 4 or 5. The matrix β is thus diagonal in the A, B, C, D, E, F-components.

According to the conclusion stated after (2.32) any admissible λ can be written in the fully contracted form:

$$\lambda = \dots \beta^k \dots \nu \dots \beta_l \dots \beta_k \dots \nu \dots \beta^l \dots \tag{4.9}$$

Furthermore, it can be deduced from (4.4) that ν is diagonal in the A, B, C, D, E, F-components. Consequently:

$$[\nu, \beta] = 0. \tag{4.10}$$

From (4.10) and (3.5)–(3.7) of [5] we have:

$$[\lambda, \beta] = 0. \tag{4.11}$$

The remaining condition (3.8) of [5] is thus satisfied, provided that

$$\lambda^{+} = -\lambda. \tag{4.12}$$

We are now able to give the simplest expression for the most general λ-term (4.9), which is the most general operator, commuting with all operators q_{kl}. Just as in [5] this can be done by an algebraic analysis of (4.9). However, the result can be foreseen immediately in the following way. Since $\tilde{\psi}\,\lambda\,\psi$ has to be an invariant, we can form all possible real invariants which can be constructed from $\tilde{\psi}$ and ψ, written in A, B, C, D, E, F-components:

$$\left.\begin{aligned} &\tilde{A}_{klm} A^{klm},\ \tilde{B}_{kl} B^{kl},\ \tilde{C}_{kl} C^{kl},\ \tilde{D}_k D^k, \\ &\tilde{E}_k E^k,\ \tilde{F} F,\ \tilde{D}_k E^k + \tilde{E}_k D^k,\ i(\tilde{D}_k E^k - \tilde{E}_k D^k). \end{aligned}\right\} \tag{4.13}$$

The most general λ-term can consequently be written ($N = A, B, C, D, E, F$):

$$\lambda = -i\hbar\,(\sum_N \varkappa_N \omega_N + \varkappa_{DE}\,\omega_{DE} + \varkappa_{DE}^{*}\,\omega_{ED}). \tag{4.14}$$

Here the projection operators ω_N are defined by the independent A, B, C, D, E, F-components of the resulting wave functions $\omega_N\,\psi$ in the following way:

$$M\text{-comp. of } \omega_N \psi = \left\{\begin{array}{l} \text{corresponding } N\text{-component of } \psi, \text{ if } M = N \\ 0, \text{ if } M \neq N \end{array}\right\}. \tag{4.15}$$

Moreover, the operators ω_{DE} and ω_{ED} are defined by

$$\left.\begin{array}{l} D_k\text{-comp. of } \omega_{ED}\psi = 0 \\ E_k\text{-comp. of } \omega_{ED}\psi = D_k \end{array}\right\}, \tag{4.16}$$

$$\left.\begin{array}{l} D_k\text{-comp. of } \omega_{DE}\psi = E_k \\ E_k\text{-comp. of } \omega_{DE}\psi = 0 \end{array}\right\}, \tag{4.17}$$

all A, B, C, F-components being zero. We have:

$$\omega_{ED} = \omega_{DE}^{+}. \tag{4.18}$$

5. The wave equation in component form

Inserting (4.3), (4.14), (4.15), (4.16), (4.17) in (2.32) and using the procedure of Section 4, we obtain the A, B, C, D, E, F-components of the wave equation:

$$\begin{aligned} &\partial_m B_{kl} - \partial_l B_{km} + \tfrac{1}{3}(\partial_r B^r{}_m \delta_{kl} - \partial_r B^r{}_l \delta_{km}) + \partial_m C_{kl} - \partial_l C_{km} - 2\,\partial_k C_{lm} + \\ &\qquad + \partial_r C^r{}_m \delta_{kl} - \partial_r C^r{}_l \delta_{km} = -i\varkappa_A A_{klm}, \end{aligned} \tag{5.1}$$

$$\tfrac{1}{2}(\partial_r A_k{}^r{}_l + \partial_r A_l{}^r{}_k) + \tfrac{1}{12}[2\,\partial_k(D_l + 3\,E_l) + 2\,\partial_l(D_k + 3\,E_k) - \\ - \partial_r(D^r + 3\,E^r)\,\delta_{kl}] = -\,i\,\varkappa_B\,B_{kl}, \qquad (5.2)$$

$$\tfrac{1}{2}\,\partial_r A^r{}_{kl} + \tfrac{1}{6}[\partial_k(D_l + 3\,E_l) - \partial_l(D_k + 3\,E_k)] = -\,i\,\varkappa_C\,C_{kl}, \qquad (5.3)$$

$$-\,\partial_r B^r{}_k - 3\,\partial_r C^r{}_k + \tfrac{3}{4}\,\partial_k F = -\,i\,(\varkappa_D D_k + \varkappa_{DE} E_k), \qquad (5.4)$$

$$-\,\partial_r B^r{}_k - 3\,\partial_r C^r{}_k - \tfrac{1}{4}\,\partial_k F = -\,i\,(\varkappa_E E_k + \varkappa^*_{DE} D_k), \qquad (5.5)$$

$$-\,\partial_r(D^r - E^r) = -\,i\,\varkappa_F\,F. \qquad (5.6)$$

It can be shown, e.g. by considerations of the rest system, that all components of the wave-function ψ satisfy the following differential equation:

$$(\partial_r\partial^r - \varkappa_A\varkappa_B)(\partial_r\partial^r - \varkappa_A\varkappa_C)\cdot\left[(1 + 2\,\alpha\,\varkappa_A)\left(\frac{1}{3} + \frac{\varkappa_B}{\varkappa_C}\right)\partial_r\partial^r - \varkappa_A\varkappa_B\right]\cdot \\ \cdot(\partial_r\partial^r - \varkappa_1^2)(\partial_r\partial^r - \varkappa_2^2)\,\psi = 0, \qquad (5.7)$$

where $\varkappa_1$ and $\varkappa_2$ are the roots of the equation

$$\begin{vmatrix} \alpha\,\varkappa^2 - \varkappa_B & \beta\,\varkappa^2 \\ \gamma\,\varkappa^2 & \delta\,\varkappa^2 - \varkappa_F \end{vmatrix} = 0, \qquad (5.8)$$

and

$$\alpha = \frac{1}{4}\,\frac{3\,\varkappa_D + \varkappa_E - \varkappa_{DE} - 3\,\varkappa^*_{DE}}{\varkappa_D\varkappa_E - \varkappa_{DE}\varkappa^*_{DE}}, \qquad (5.9)$$

$$\beta = -\,\frac{1}{4}\,\frac{3\,\varkappa_D - 3\,\varkappa_E - \varkappa_{DE} + 9\,\varkappa^*_{DE}}{\varkappa_D\varkappa_E - \varkappa_{DE}\varkappa^*_{DE}}, \qquad (5.10)$$

$$\gamma = -\,\frac{1}{4}\,\frac{\varkappa_D - \varkappa_E + \varkappa_{DE} - \varkappa^*_{DE}}{\varkappa_D\varkappa_E - \varkappa_{DE}\varkappa^*_{DE}}, \qquad (5.11)$$

$$\delta = \frac{1}{4}\,\frac{\varkappa_D + 3\,\varkappa_E + \varkappa_{DE} + 3\,\varkappa^*_{DE}}{\varkappa_D\varkappa_E - \varkappa_{DE}\varkappa^*_{DE}}. \qquad (5.12)$$

The scheme gives evidently a wide possibility of varying the ratios between the five mass eigenvalues of (5.7). As a special case unique mass is obtained by putting

$$\varkappa_C = \varkappa_B\,, \qquad (5.13)$$

$$\varkappa_1^2 = \varkappa_2^2 = \varkappa_A\varkappa_B \qquad (5.14)$$

and either

$$\alpha = -\,\frac{1}{2\,\varkappa_A} \qquad (5.15)$$

or

$$\alpha = -\,\frac{1}{8\,\varkappa_A} \qquad (5.16)$$

The unique mass case (5.15) is e.g. afforded by the following choice of constants:

$$\left.\begin{aligned} \varkappa_D &= -\tfrac{1}{2}\varkappa_A, \quad \varkappa_E = \tfrac{25}{6}\varkappa_A, \\ \varkappa_F &= -\tfrac{3}{7}\varkappa_B, \quad \varkappa_{DE} = \varkappa_{DE}^* = -\tfrac{1}{2}\varkappa_A, \end{aligned}\right\} \tag{5.17}$$

whereas the unique mass case (5.16) is e.g. afforded by the choice

$$\left.\begin{aligned} \varkappa_D &= -2\varkappa_A, \quad \varkappa_E = \tfrac{37}{6}\varkappa_A, \\ \varkappa_F &= -\tfrac{1}{49}\varkappa_B, \quad \varkappa_{DE} = \varkappa_{DE}^* = -2\varkappa_A, \end{aligned}\right\} \tag{5.18}$$

$\varkappa_A$ and $\varkappa_B$ being in both cases arbitrary constants.

It seems also possible to obtain unique mass by choosing the constants so as to make some of the masses in (5.7) complex and the remaining masses equal.

As stated in Section 2 the 40-component wave equation (2.32) is irreducible, if λ is sufficiently general. However, with the special λ, given by

$$\varkappa_{DE} = 0, \tag{5.19}$$

$$\varkappa_D = \varkappa_E, \tag{5.20}$$

we obtain from (5.4) and (5.5):

$$-4\,\partial_r(B^r{}_k - 3\,C^r{}_k) = -i\varkappa_D(D_k + 3\,E_k), \tag{5.21}$$

$$\partial_k F = -i\varkappa_D(D_k - E_k). \tag{5.22}$$

The system of equations (5.1)–(5.6) is thus reduced into two independent systems, (5.1), (5.2), (5.3), (5.21) and (5.6), (5.22), each system containg only one independent vector, viz. $D_k + 3\,E_k$ and $D_k - E_k$ respectively. The systems contain 35 and 5 independent components corresponding to the cases $R_5(2,1)$ and $R_5(1,0)$ in the formalism of Bhabha [3]. However, they are somewhat more general than the corresponding case of Bhabha. In fact, Bhabha assumes the mass term to be a constant times the unit matrix, whereas we have chosen the mass term as general as possible. If we make Bhabha's assumption, i.e. put

$$\varkappa_A = \varkappa_B = \varkappa_C = \varkappa_D = \varkappa_E = \varkappa_F = \varkappa, \tag{5.23}$$

all components of the wave function are seen to satisfy

$$(\partial_r\partial^r - \varkappa^2)\left(\partial_r\partial^r - \frac{\varkappa^2}{4}\right)\psi = 0, \tag{5.24}$$

as is required by the Bhabha scheme.

6. The gravitational zero mass limit

Before discussing the zero mass limit, we have to make a slight generalization of the Dirac equation (2.32) in the following way:

$$[(\Lambda\beta^k\Lambda)\,p_k - \lambda]\psi = 0, \tag{6.1}$$

where Λ is an expression of the form (4.14), but Hermitian:

$$\Lambda = \sum_N k_N \omega_N + k_{DE}\,\omega_{DE} + k^*_{DE}\,\omega_{ED}\,. \tag{6.2}$$

It is evident that if Λ is non-singular, (6.1) can be transformed to the form (2.32). It is thus no reason for assuming a non-singular Λ to be anything but the unit matrix.

If Λ is singular, (6.1) can be transformed into an equation of the same form as (6.1) but with a Λ having no other eigenvalues than 0 or 1. We can thus assume that a singular Λ has this property.

The reduction, discussed at the end of Section 5 and a following selection of the case $R_5(2,1)$ can now be performed by choosing

$$k_F = k_{DE} = 0\,, \tag{6.3}$$

$$\varkappa_{DE} = 0\,, \quad \varkappa_D = \varkappa_E\,. \tag{6.4}$$

In fact, $k_F = 0$ means the vanishing of the left hand side of (5.6) and the disappearance of F from (5.4) and (5.5). From (5.4) and (5.5) then follows

$$D_k = E_k \tag{6.5}$$

and from (5.6):

$$F = 0. \tag{6.6}$$

(3.16), (3.17), (3.18), (6.5) and (6.6) then give

$$\psi_\alpha{}^{\alpha\gamma} = 0\,, \tag{6.7}$$

which is the condition to be imposed on the affine tensor $\psi^{\alpha\beta\gamma}$ in order to obtain the case $R_5(2,1)$.

In order to obtain Einsteins linearized equations for the gravitational field we have to make another choice of the arbitrary constants, viz.

$$k_C = k_E = k_{DE} = 0\,, \tag{6.8}$$

$$\varkappa_A = 2\,, \quad \varkappa_D = -1\,, \tag{6.9}$$

$$\varkappa_B = \varkappa_C = \varkappa_E = \varkappa_F = \varkappa_{DE} = 0\,. \tag{6.10}$$

The choice (6.8), (6.10) means that the equations (5.3) and (5.5) disappear and that the antisymmetric tensor C_{kl} and the vector E_k disappear from the remaining equations (5.1), (5.2), (5.4), (5.6). By this procedure only 30 independent components remain, so that the matrices of the Dirac equation are now as a matter of fact 30×30-rowed.

In order to obtain connexion with the linearized theory of gravitation we introduce the following fields:

$$V_{klm} = i\,\psi_{klm} = i\left[A_{klm} + \tfrac{1}{3}\left(D_m\,\delta_{kl} - D_l\,\delta_{km}\right)\right], \tag{6.11}$$

$$h_{kl} = \tfrac{1}{2}\left(\psi_{kl5} + \psi_{lk5}\right) = B_{kl} + \tfrac{1}{4}\,F\,\delta_{kl}\,. \tag{6.12}$$

Now (5.1) and (5.4) give:

$$V_{klm} = \tfrac{1}{2}(\partial_l h_{mk} - \partial_m h_{lk} + \partial_m h_r{}^r \delta_{kl} - \partial_r h_m{}^r \delta_{kl} - \partial_l h_r{}^r \delta_{km} + \partial_r h_l{}^r \delta_{km}) =$$
$$= \tfrac{1}{2} \delta_{[s}^{\,r} \delta_l^k \delta_{m]}^t \partial_t h_r{}^s, \quad (6.13)$$

where [] denotes the sum of the even permutations minus the sum of the odd permutations of the indices between the brackets.

Moreover (5.2) and (5.6) give:

$$\partial_r (V_k{}^r{}_l + V_l{}^r{}_k) = 0. \quad (6.14)$$

Since by (6.13)

$$\partial_r V_k{}^r{}_l = \partial_r V_l{}^r{}_k, \quad (6.15)$$

(6.14) can also be written

$$\partial_r V_k{}^r{}_l = 0. \quad (6.16)$$

If we put

$$g_{kl} = \delta_{kl} + \varepsilon h_{kl} + \cdots, \quad (6.17)$$

the expression (6.13) is seen to give V_{klm} as the gravitational superpotential of Freud [13], from which $G_{kl} = R_{kl} - \frac{1}{2} R \delta_{kl}$ can be derived by means of

$$G_{kl} = \partial_r V_k{}^r{}_l. \quad (6.18)$$

The equation (6.16) now gives the linearized Einstein equations for the gravitational field in a general gauge

$$G_{kl} = 0, \quad (6.19)$$

or

$$\partial_r \partial^r h_{kl} + \partial_k \partial_l h_r{}^r - \partial_k \partial_r h_l{}^r - \partial_l \partial_r h_k{}^r - (\partial_r \partial^r h_s{}^s - \partial_r \partial_s h^{rs}) \delta_{kl} = 0. \quad (6.20)$$

Institute of Theoretical Physics, University of Stockholm, and Division of Mechanics II, Royal Institute of Technology, Stockholm, January 1959.

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